Math 2300: Calculus II, Fall 2005; Instructor: Dana Ernst
Partial Solutions for Midterm Exam 1

Here are the solutions for Midterm Exam 1. In some cases, I haven't shown all the work, but rather pointed you in the right direction.

1. On these problems, there's no calculus required. All we need to do is evaluate each expression using the appropriate definition.

   (a) Here, work your way from the inside out:

   $$\cos(\sin^{-1}(\sqrt{3}/2)) = \cos(\pi/3) = 1/2.$$

   (b) Recall that cosine has period 2π, so that we get

   $$\sec 13\pi/6 = \frac{1}{\cos 13\pi/6} = \frac{1}{\cos \pi/6} = \frac{1}{\sqrt{3}/2} = 2/\sqrt{3}.$$

   (c) We need to use the definition of hyperbolic tangent and then simplify.

   $$\tanh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{e^{\ln 5} + e^{-\ln 5}} = \frac{5 - 5^{-1}}{5 + 5^{-1}} = 12/13.$$

2. These questions are testing whether we know how to apply the Chain Rule.

   (a) On this one, it is best to use the properties of logs to rewrite the expression before differentiating.

   $$y = \ln\sqrt{\frac{x-1}{x+1}} = \frac{1}{2}(\ln(x-1) - \ln(x+1)).$$

   Then we get

   $$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right).$$

   (b) On this one, just apply the chain rule:

   $$\frac{dy}{dx} = e^{\arctan(4x)} \cdot \frac{1}{1+(4x)^2} \cdot 4.$$

3. These three integrals are very similar. The first can be done with straight u-substitution and the last two can be made to resemble integration formulas.

   (a) Let $u = 4 - x^4$. Then $du = -4x^3 dx$, and so, we get

$$
\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^4}}\,dx &= \int \frac{x^3}{u^{1/2}} \cdot \frac{du}{-4x^3} \\
&= -\frac{1}{4}\int u^{-1/2}\,du \\
&= -\frac{1}{2}u^{1/2} + C \\
&= -\frac{1}{2}\sqrt{4-x^4} + C.
\end{aligned}
$$

(b) Let $u = x^2$. Then $du = 2xdx$. Looking at Formula #19, we see that

$$
\begin{aligned}
\int \frac{1}{x\sqrt{x^4-4}}\,dx &= \frac{1}{2}\int \frac{1}{u\sqrt{u^2-4}}\,du \\
&= \frac{1}{4}\text{arcsec}\left|\frac{\text{x}^2}{2}\right| + \text{C}.
\end{aligned}
$$

(c) Again, let $u = x^2$. Looking at Formula #45, we see that

$$
\begin{aligned}
\int \frac{x}{\sqrt{x^4-4}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{u^2-4}}\,du \\
&= \frac{1}{2}\ln\left|u + \sqrt{u^2-4}\right| + C \\
&= \frac{1}{2}\ln\left|x^2 + \sqrt{x^4-4}\right| + C.
\end{aligned}
$$

4. The first of these uses integration by parts. The second one uses straight u-substitution. The last one also uses u-substitution, but we first need to rewrite it in a clever way.

(a) Let $u = \ln x$ and $dv = x\ dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$. So, we get

$$
\begin{aligned}
\int x \ln x\ dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx \\
&= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\ dx \\
&= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.
\end{aligned}
$$

(b) Let $u = \tan x$. Then $du = \sec^2 x\ dx$, and so, we get

$$
\begin{aligned}
\int \sec^2 x \tan x\ dx &= \int u\ du \\
&= \frac{u^2}{2} + C \\
&= \frac{1}{2}\tan^2 x + C.
\end{aligned}
$$

(c) Here, we save a factor of sine and change the remaining factors into cosines using the Pythagorean Identity. Then let $u = \cos x$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^4 x \sin^3 x\ dx &= \int_0^{\pi/2} \cos^4 x(1-\cos^2 x)\sin x\ dx \\
&= -\int_{x=0}^{x=\pi/2} u^4(1-u^2)\ du \\
&= -\int_{x=0}^{x=\pi/2} u^4 - u^6\ du \\
&= -\left(\frac{\cos^5 x}{5} - \frac{\cos^7 x}{7}\right)_0^{\pi/2} \\
&= \frac{2}{35}.
\end{aligned}$$

5. Here is the easiest way to do this problem. First, note that since the doubling time is 4 hours, we get

$$\begin{aligned}
T_{double} &= \frac{\ln 2}{r} \\
4 &= \frac{\ln 2}{r} \\
r &= \frac{\ln 2}{4}.
\end{aligned}$$

Then plug all of the necessary information into the equation $A = Pe^{rt}$. To solve for t, we need to take the natural log of both sides of the equation:

$$\begin{aligned}
50 &= e^{\frac{\ln 2}{4}t} \\
\ln 50 &= \ln e^{\frac{\ln 2}{4}t} \\
\ln 50 &= \frac{\ln 2}{4}t \\
t &= \frac{4\ln 50}{\ln 2}\ \text{hours.}
\end{aligned}$$

6. Use integration by parts. Let $u = \sec x$ and $dv = \sec^2 x\ dx$. Then $du = \sec x \tan x\ dx$ and $v = \tan x$. Then we get

$$\begin{aligned}
\int \sec^3 x\ dx &= \sec x \tan x - \int \sec x \tan^2 x\ dx \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1\ dx \\
&= \sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx.
\end{aligned}$$

But then

$$\begin{aligned}
2\int \sec^3 x \, dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\int \sec^3 x \, dx &= \frac{1}{2}\left[\ln|\sec x + \tan x|\right] + C.
\end{aligned}$$

7. First, notice that

$$f(x) = \int_x^1 \frac{1}{\sqrt{1-t^2}}\, dt = -\int_1^x \frac{1}{\sqrt{1-t^2}}\, dt.$$

(a) By the Fundamental Theorem of Calculus, we have

$$\frac{d}{dx} f(x) = -\frac{1}{\sqrt{1-x^2}}.$$

(b) This was the hard part. Using implicit differentiation (with respect to x), we get

$$\begin{aligned}
\frac{d}{dx}[x] &= \frac{d}{dx}[g(y)] \\
1 &= g'(y)\cdot\frac{dy}{dx} \qquad \text{(Chain Rule)} \\
g'(y) &= \frac{1}{dy/dx} \\
g'(y) &= -\sqrt{1-x^2} \qquad \left(\frac{dy}{dx} = f'(x) = -\frac{1}{\sqrt{1-x^2}}\right) \\
g'(y) &= -\sqrt{1-g(y)^2} \qquad (x = g(y)).
\end{aligned}$$

(c) Recall that

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}.$$

This implies that $f(x) = \arccos x$ (plus a constant perhaps). Since g is the inverse of f, we must have

$$g(x) = \cos x.$$

Notice that the answer in part (b) agrees with this. That is, if $g(x) = \cos x$ and $g'(x) = -\sqrt{1-g(x)^2}$, then

$$g'(x) = -\sqrt{1-\cos^2 x} = -\sqrt{\sin^2 x} = -\sin x,$$

which really is the derivative of cosine. Also, note that all of the discussion at the beginning of the problem is indicating to us why we refer to inverse cosine as arccos (arc length!).