

## Section 12.1: Sequences

### Goal

In this section, we introduce sequences, including some terminology, and look at several examples.

### Definition of a sequence and basic examples

**Definition 1.** A *sequence* is a list (usually infinite) of objects (usually numbers, but does not have to be) that has a specified order. More specifically, a sequence is a function with domain  $\mathbb{N}$ .

We usually denote the *terms* of a sequence with subscripts:

$$a_1, \underbrace{a_2}_{2\text{nd}}, a_3, \dots, \underbrace{a_n}_{n\text{th}}, \underbrace{a_{n+1}}_{(n+1)\text{th}}, \dots$$

We denote the entire sequence by  $\{a_n\}_{n=1}^{\infty}$  (or possibly  $(a_n)_{n=1}^{\infty}$ ) or simply  $\{a_n\}$  (respectively,  $(a_n)$ ).

#### Example 2.

- (a)  $\{a_n\} = \{1, 2, 3, 4, \dots\}$ . In this case,  $a_n =$ \_\_\_\_\_ . Also:

$$a_3 = \underline{\hspace{2cm}}$$

$$a_{16} = \underline{\hspace{2cm}}$$

- (b)  $\{b_k\} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ . In this case,  $b_k =$ \_\_\_\_\_ . Also:

$$b_4 = \underline{\hspace{2cm}}$$

$$b_7 = \underline{\hspace{2cm}}$$

- (c)  $\{c_n\} = \{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125}, \dots\}$ . In this case,  $c_n =$ \_\_\_\_\_ . Also:

$$c_3 = \underline{\hspace{2cm}}$$

$$c_{12} = \underline{\hspace{2cm}}$$

- (d)  $\{a_i\} = \{1, 1, 1, \dots\}$ . In this case,  $a_i =$ \_\_\_\_\_ . This is an example of a \_\_\_\_\_ .  
Also:

$$a_3 = \underline{\hspace{2cm}}$$

$$a_{100} = \underline{\hspace{2cm}}$$

- (e)  $\{p_n\} = \{2, 3, 5, 7, 11, 13, 17, \dots\}$  (sequence of prime numbers). In this case, there is no formula for  $p_n$ . However, we can find:

$$p_3 = \underline{\hspace{2cm}}$$

$$p_7 = \underline{\hspace{2cm}}$$

- (f)  $\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$  (called the *Fibonacci sequence*). This is an example of a *recursive sequence*. In this case,  $f_1 =$ \_\_\_\_\_,  $f_2 =$ \_\_\_\_\_, and  $f_n =$ \_\_\_\_\_ for  $n \geq$ \_\_\_\_\_. Also:

$$f_6 = \underline{\hspace{2cm}}$$

$$f_{11} = \underline{\hspace{2cm}}$$

**Note 3.** Sometimes we will begin a sequence at a different index other than 1.

**Example 4.**

(a) Consider the sequence  $\{c_n\}$  in Example 2(c). What is  $\{c_n\}_{n=3}^{\infty}$ ?

(b) Consider the sequence  $\{b_k\}$  in Example 2(b) and compare with  $\{b'_k\}_{k=0}^{\infty}$ , where  $b'_k = \frac{1}{k+2}$ .

We can draw “graphs” of sequences, where the  $x$ -axis is  $\mathbb{N}$  and the  $y$ -axis is the set of values that the sequence takes (for us, the  $y$ -axis will usually be  $\mathbb{R}$ ).

**Example 5.** Draw a graph of the sequence  $\left\{\frac{(-1)^{n+1}}{n}\right\}$ .

## Limits of sequences

**Definition 6.** A sequence  $\{a_n\}$  has the *limit*  $L$  and we write

$$\lim_{n \rightarrow \infty} a_n = L$$

if we can make the terms of  $\{a_n\}$  as close to  $L$  as we like by taking  $n$  sufficiently large. If such a  $L$  exists, we say that the sequence                     . Otherwise, we say that the sequence                     .

**The Picture:**

**Theorem 7.** *If  $\lim_{n \rightarrow \infty} f(x) = L$  and  $f(n) = a_n$ , then  $\lim_{n \rightarrow \infty} a_n = L$ , as well.*

**Important Note 8.** What this means is that we get to use all of our previous limit weapons (limit laws, Squeeze Theorem, L'Hospital's Rule, etc.).

Here is another weapon.

**Theorem 9.** *If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n =$ \_\_\_\_\_.*

**The Picture:**

**Example 10.** Converge or diverge? If the sequence converges, find its limit.

(a)  $\left\{ \frac{1}{n} \right\}$

(b)  $\left\{ \frac{3n^2 + 2n + 2}{1 - n^2} \right\}$

(c)  $\{(-1)^n\}$

(d)  $\left\{\frac{(-1)^n}{n^2+1}\right\}$

(e)  $\left\{\frac{\ln n}{n}\right\}$

(f)  $\{\sin(\pi/n)\}$

(g)  $\left\{\frac{n!}{n^n}\right\}$  (Hint: use Squeeze Theorem)

## More terminology

### Definition 11.

1.  $\{a_n\}$  is *increasing* if \_\_\_\_\_ for all  $n \geq 1$ .
2.  $\{a_n\}$  is *decreasing* if \_\_\_\_\_ for all  $n \geq 1$ .
3.  $\{a_n\}$  is *monotonic* if it is either increasing or decreasing.
4.  $\{a_n\}$  is *bounded above* (respectively, *below*) if there exists  $M \in \mathbb{R}$  such that \_\_\_\_\_ (respectively, \_\_\_\_\_).

**Example 12.** Show that  $\left\{ \frac{n}{n^2 + 1} \right\}$  is decreasing.

**Theorem 13** (Monotonic Sequence Theorem). *Every bounded (above and below) monotonic sequence is convergent.*