Introduction to Linear Algebra, Spring 2007
MATH 3130, Section 001

SOLUTIONS TO EXAM 2

YOUR NAME: SOLUTIONS

Instructions: Answer each of the following questions completely. To receive full credit, you must *justify* each of your answers (unless stated otherwise). How you reached your answer is more important than the answer itself. If something is unclear, or if you have any questions, then please ask. Good luck!

1. Let A be a matrix that is row equivalent to one of the following matrices in echelon form.

$$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

For each of the following statements, state whether A is row equivalent to M_1, M_2, or M_3. You should write either M_1, M_2, or M_3 in each blank provided. You do *not* need to justify your answers. [**5 points each**]

$\underline{M_1}$ A is invertible. [**IMT**]

$\underline{M_2}$ If $T(\mathbf{x}) = A\mathbf{x}$, then T is neither one-to-one or onto.[**IMT**]

$\underline{M_3}$ If $T(\mathbf{x}) = A\mathbf{x}$, then T is onto, but not one-to-one. [**Theorem 1.12**]

$\underline{M_2}$ A^T has the same size as A, but is not invertible. [**IMT**]

$\underline{M_1}$ $\det A \neq 0$. [**IMT**]

$\underline{M_3}$ The columns of A are linearly dependent, but span all of $\mathbb{R}^3$.
[**Blue Box on pg 66, Theorem 1.4**]

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points through the vertical x_2-axis and then rotates points $\pi/2$ radians (counterclockwise).

   (a) Find the standard matrix for this linear transformation. **[12 points]**

   Solution: If you reflect $(1, 0)$ and $(0, 1)$ through the vertical x_2-axis, you get $(-1, 0)$ and $(0, 1)$, respectively. If you rotate these points $\pi/2$ radians, you get $(0, -1)$ and $(-1, 0)$, respectively. That is, $T(\mathbf{e}_1) = (0, -1)$ and $T(\mathbf{e}_2) = (-1, 0)$. Therefore, the standard matrix for T is

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

   (b) What single geometric transformation is T equivalent to? You must justify your answer. **[10 points]**

   Solution: Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $x_1 = -x_2$. Then $S(\mathbf{e}_1) = (0, -1)$ and $S(\mathbf{e}_2) = (-1, 0)$. Therefore, $S = T$.

3. Find the *LU-factorization* for the following matrix. [**12 points**]

$$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix}.$$

Solution: Recall that the only row operation that we are allowed to use for our algorithm is row replacement. Using only row replacement, row reduce A to echelon form:

$$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Therefore, we have

$$U = \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix}.$$

To get the first column of L, divide the first column of A by 3. To get the second column of L, divide the last two entries in the second column of the middle matrix above by 5. We get

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1/3 & 1 & 1 \end{bmatrix}.$$

4. Calculate the determinant of the following matrix. [**12 points**]

$$A = \begin{bmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{bmatrix}.$$

Solution: There are lots of ways to do this problem. Here's one way. Row reduce A to echelon form. Along the way, keep track of any changes in the determinant. For example, if we swap any rows, we pick up a negative sign. When we're done, we can just multiply the entries along the diagonal (and account for any scalars or sign changes).

$$\begin{aligned} |A| &= -\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 1 & 5 & 5 \\ 0 & 2 & 7 & 3 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -5 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= -3. \end{aligned}$$

5. The following statement is *FALSE* in general:

If A and B are $n \times n$ matrices, then $(A+B)(A-B) = A^2 - B^2$.

Provide a counterexample where A and B are 2×2 matrices and justify your answer by calculating both sides of the equation above. [**12 points**]

Solution: You want to provide a counterexample to show that the above statement is false. There are many possible examples. If you choose diagonal, upper triangular, or lower triangular, you might run into trouble (that is, your example might not be a counterexample). How about these? Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 2 & 3 \\ 5 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix},$$

which implies that

$$(A+B)(A-B) = \begin{bmatrix} 3 & 11 \\ 5 & 20 \end{bmatrix}.$$

On the other hand, we see that

$$A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \quad \text{and} \quad B^2 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix},$$

which implies that

$$A^2 - B^2 = \begin{bmatrix} 4 & 8 \\ 11 & 19 \end{bmatrix}.$$

Lastly, we see that

$$(A+B)(A-B) = \begin{bmatrix} 3 & 11 \\ 5 & 20 \end{bmatrix} \neq \begin{bmatrix} 4 & 8 \\ 11 & 19 \end{bmatrix} = A^2 - B^2.$$

6. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $H = \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\}$. Prove that H is a subspace of $\mathbb{R}^n$. **[12 points]**

Solution: Note that T is not necessarily the 0-transformation. H is the collection of all vectors in $\mathbb{R}^n$ that get mapped to $\mathbf{0} \in \mathbb{R}^m$, which doesn't have to be all of $\mathbb{R}^n$. We have three things to show in order for H to be a subspace.

(i) Since $T(\mathbf{0}) = \mathbf{0}$ for all linear transformations, $\mathbf{0} \in H$.

(ii) Let $\mathbf{u}, \mathbf{v} \in H$. Then $T(\mathbf{u}) = \mathbf{0}$ and $T(\mathbf{v}) = \mathbf{0}$. This implies that

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

since T is linear. Therefore, $\mathbf{u} + \mathbf{v} \in H$.

(iii) Let $\mathbf{u} \in H$ and $c \in \mathbb{R}$. Then $T(\mathbf{u}) = \mathbf{0}$, which implies that

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0},$$

since T is linear. Therefore, $c\mathbf{u} \in H$.