

**Introduction to Linear Algebra, Spring 2007**  
**MATH 3130, Section 001**  
**SOLUTIONS TO PROJECT 2**

1. Prove Theorem 2.3(d).

**Proof:** Let  $A = [a_{ij}]$  and  $B = [b_{ij}]$  be  $n \times n$  matrices. We will show that the  $(i, j)$ -entry of  $(AB)^T$  equals the  $(i, j)$ -entry of  $B^T A^T$  for arbitrary  $1 \leq i, j \leq n$ . The  $(i, j)$ -entry of  $(AB)^T$  is the  $(j, i)$ -entry of  $AB$  (by definition). The  $(j, i)$ -entry of  $AB$  is the dot product of the  $j$ th row of  $A$  with the  $i$ th column of  $B$  (by definition):

$$(a_{j1}, \dots, a_{jn}) \cdot (b_{1i}, \dots, b_{ni}) = a_{j1}b_{1i} + \dots + a_{jn}b_{ni}.$$

On the other hand, the  $(i, j)$ -entry of  $B^T A^T$  is the dot product of the  $i$ th row of  $B^T$  with the  $j$ th column of  $A^T$ . But the  $i$ th row of  $B^T$  is the  $i$ th column of  $B$ , and the  $j$ th column of  $A^T$  is the  $j$ th row of  $A$ . So, the  $(i, j)$ -entry of  $B^T A^T$  is

$$(b_{1i}, \dots, b_{ni}) \cdot (a_{j1}, \dots, a_{jn}) = a_{j1}b_{1i} + \dots + a_{jn}b_{ni},$$

which is the same as above. Therefore,  $(AB)^T = B^T A^T$ .

2. Let  $V$  be a vector space (over  $\mathbb{R}$ ). Let  $H$  and  $K$  be subspaces of  $V$ . The intersection of  $H$  and  $K$ , written as  $H \cap K$ , is the set of all  $\mathbf{v} \in V$  that belong to both  $H$  and  $K$ . Prove that  $H \cap K$  is a subspace of  $V$ .

**Proof:** We have three things to check.

- (i) Since  $H$  and  $K$  are subspaces,  $\mathbf{0} \in H$  and  $\mathbf{0} \in K$ . Therefore,  $\mathbf{0} \in H \cap K$ .
- (ii) Let  $\mathbf{u}, \mathbf{v} \in H \cap K$ . Then  $\mathbf{u}, \mathbf{v} \in H$  and  $\mathbf{u}, \mathbf{v} \in K$ . Since  $H$  and  $K$  are subspaces,  $\mathbf{u} + \mathbf{v} \in H$  and  $\mathbf{u} + \mathbf{v} \in K$ . Therefore,  $\mathbf{u} + \mathbf{v} \in H \cap K$ .
- (iii) Let  $\mathbf{u} \in H \cap K$  and  $c \in \mathbb{R}$ . Then  $\mathbf{u} \in H$  and  $\mathbf{u} \in K$ . Since  $H$  and  $K$  are subspaces,  $c\mathbf{u} \in H$  and  $c\mathbf{u} \in K$ . Therefore,  $c\mathbf{u} \in H \cap K$ .

Since (i), (ii), and (iii) hold,  $H \cap K$  is a subspace of  $V$ .

3. Let  $V = \mathbb{R}^2$  with the usual scalar multiplication, but instead of the usual definition of addition, define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1).$$

Determine whether  $V$  is a vector space. If  $V$  is a vector space, then prove it. If  $V$  is not a vector space, then show that one of the axioms of vector spaces fails.

**Solution:**  $V$  is *not* a vector space. At first glance, you might think that  $V$  is not a vector space because it doesn't appear to have a vector that acts like the zero vector. But in fact,  $(-1, -1)$  plays the role of the zero vector. One of the axioms of the definition of vector space that does fail is Axiom 7. Let  $c = d = 1$  and let  $\mathbf{u} = (1, 1)$ . Then

$$(c + d)\mathbf{u} = (1 + 1)(1, 1) = 2(1, 1) = (2, 2).$$

On the other hand,

$$c\mathbf{u} + d\mathbf{u} = 1(1, 1) + 1(1, 1) = (1, 1) + (1, 1) = (2 + 1, 2 + 1) = (3, 3).$$

Since  $(2, 2) \neq (3, 3)$ , Axiom 7 fails. Therefore,  $V$  is not a vector space.

4. Read Section 1.10 and complete Exercise 10 on page 101.

**Solution:** For this problem, the migration matrix is

$$M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}.$$

In 2000, we are given that

$$\mathbf{x}_0 = \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix}.$$

Then in 2002, we have

$$\mathbf{x}_2 = M^2\mathbf{x}_0 = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix} \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix} \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix} = \begin{bmatrix} 722,100 \\ 577,900 \end{bmatrix}.$$

Therefore, there will be 722,100 people in the city and 577,900 people in the suburbs in 2002.

5. Read Section 2.7 and complete Exercise 18 on page 166.

**Solution:** Let  $T_1$  be the (linear) transformation that rotates points in  $\mathbb{R}^3$  about the  $z$ -axis through an angle of  $-30^\circ$ . Then  $T_1(\mathbf{e}_1) = (\cos -30^\circ, \sin -30^\circ)$ ,  $T_1(\mathbf{e}_2) = (\cos 60^\circ, \sin 60^\circ)$ , and  $T_1(\mathbf{e}_3) = \mathbf{e}_3$ . Then the matrix for this rotation for homogeneous coordinates is

$$\begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the translation, we want  $(x, y, z, 1)$  to map to  $(x + 5, y - 2, z + 1, 1)$ . The matrix that does this is

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the matrix that first rotates and then translates is

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$