Section 8.7: Approximate Integration

Goal

In this section, we will discuss several methods for approximating definite integrals.

Introduction

There are two situations in which it is impossible to find the exact value of a definite integral.

1. In order to evaluate
$$\int_a^b f(x)\ dx$$
using the FTC, we need to know an antiderivative of f. However, sometimes this is difficult, or even impossible.

2. Sometimes when data is collected (like in a scientific experiment) all we have is discrete data points. In this case, we don't have a formula for the corresponding function.

Let's introduce a few approximation techniques and then we'll do some examples.

Left/Right endpoint approximation

Recall that the definite integral on $[a, b]$ is defined in terms of Riemann sums. If we divide $[a, b]$ into n equal width subintervals, then
$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$
where x_i^* is any point in the ith subinterval.

Here's the picture:

If we take x_i^* to be the left endpoint of the subinterval, then $x_i^* = x_{i-1}$. In this case, we can approximate the integral using
$$\boxed{\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x = L_n.}$$

Here's the picture:

Similarly, if we take x_i^* to be the right endpoint of the subinterval, then $x_i^* = x_1$. In this case, we can approximate the integral using

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(x_i)\Delta x = R_n.$$

Midpoint rule

If we take x_i^* to be the midpoint of the subinterval, then $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$. In this case, we can approximate the integral using

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f\left(\frac{1}{2}(x_{i-1} + x_i)\right)\Delta x = M_n.$$

Trapezoid rule

There's no reason why we have to use rectangles to approximate. Instead, let's approximate the shape of the function using straight line segments and approximate the area under the curve using the corresponding trapezoids.

Here's the picture:

The area of each trapezoid is Δx times the average of the height at the left endpoint with the height at the right endpoint. That is, the average of the ith trapezoid is

$$\frac{\Delta x}{2}\left(f(x_{i-1})+f(x_i)\right).$$

So, to approximate the integral using trapezoids use

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right)=T_n.$$

Simpson's rule

Well, why not use curves to approximate curves? We can just as well use parabolas to approximate the shape of the curve.

Here's the picture:

The book has a very nice presentation on how to determine the area of each of the corresponding shapes. Here is Simpson's rule for approximating the area under a curve

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)=S_n,$$

where we require n to be *even*. The patten of the coefficients is $1,4,2,4,\ldots,2,4,1$.

Comment

Of course, for each of the techniques, the larger n is, the better the approximation.

Examples

OK, let's do some examples.

Example 1. Consider $\int_0^1 e^{x^2}\ dx$. Approximate this integral using each of the approximation techniques.

1. Midpoint (with $n = 5$):

2. Trapezoid (with $n = 5$):

3. Simpson's (with $n = 6$):

Example 2. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see table). Use Simpson's rule to estimate the distance (in meters) the runner covered during those 5 seconds.

t (s)	v (m/s)
0.0	0
0.5	4.67
1.0	7.34
1.5	8.86
2.0	9.73
2.5	10.22
3.0	10.51
3.5	10.67
4.0	10.76
4.5	10.81
5.0	10.81