Solution to Exercise 2.5.6(d)

Let α be the positive soln and β the negative soln to $x^2 = x + 1$. In this case,

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then for all $n \in \mathbb{N}$,

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

<u>Pf.</u> Let α and β be as above. ~~and then~~ We will proceed by induction (PCI).

(i) <u>Basic Step:</u> For $n=1$, we see that

$$f_1 = 1$$

and

$$\frac{\alpha^1 - \beta^1}{\alpha - \beta} = 1.$$

Also, for $n=2$, we see that

$$f_2 = 1$$

and

$$\frac{\alpha^2 - \beta^2}{\alpha - \beta} = \frac{(\alpha+\beta)\cancel{(\alpha-\beta)}}{\cancel{\alpha-\beta}} = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = \frac{2}{2} = 1,$$

as desired.

(ii) <u>Inductive Step</u>: Let $n \in \mathbb{N}$ and assume that

$$f_k = \frac{\alpha^k - \beta^k}{\alpha - \beta}$$

for all $1 \le k \le n-1$. Now, we need to show that

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

We see that

$$f_n = f_{n-1} + f_{n-2} \qquad \text{(by def of Fibonacci sequence)}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} \qquad \text{(by ind hypothesis)}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1} + \alpha^{n-2} - \beta^{n-2}}{\alpha - \beta}$$

$$= \frac{\alpha^{n-2}(\alpha + 1) + -\beta^{n-2}(\beta + 1)}{\alpha - \beta}$$

$$= \frac{\alpha^{n-2}(\alpha^2) - \beta^{n-2}(\beta^2)}{\alpha - \beta} \qquad \text{(since } \alpha \,\&\, \beta \text{ are solns to } x^2 = x + 1\text{)}$$

$$= \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

(iii) By the PCI, we have our desired result. $\boxtimes$